Feynman's Ratchet and Pawl

(Lectures on Physics, Vol. 1, Chapter 46)

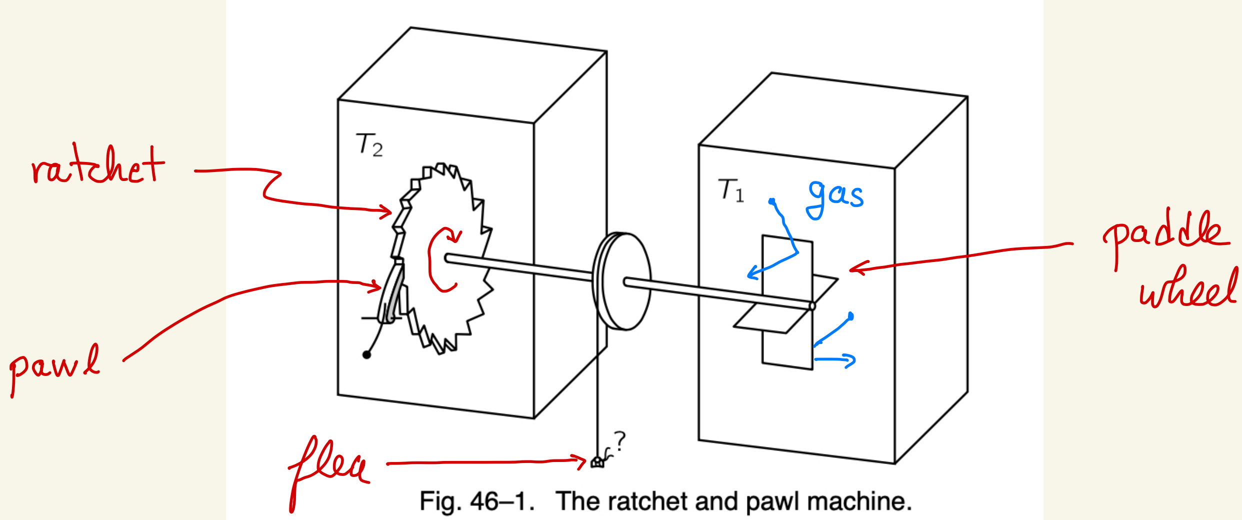

Fig. 46–1. The ratchet and pawl machine.

First assume $T_1 = T_2 = T$.

Argument:

- The ratchet can only rotate clockwise (CW), as the pawl blocks CCW rotations.
- The random collisions of gas particles against the paddles (thermal noise) will occasionally produce CW rotation.
- In the long run, the flea is lifted.

* Violation of 2nd Law (Kelvin-Planck)

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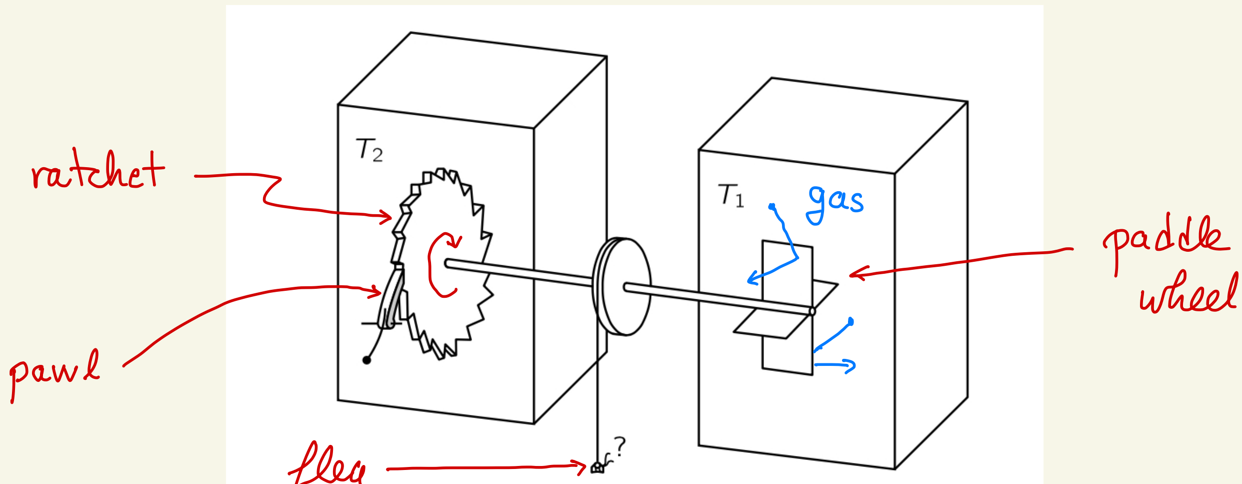


Fig. 46–1. The ratchet and pawl machine.

Resolution of the apparent paradox:

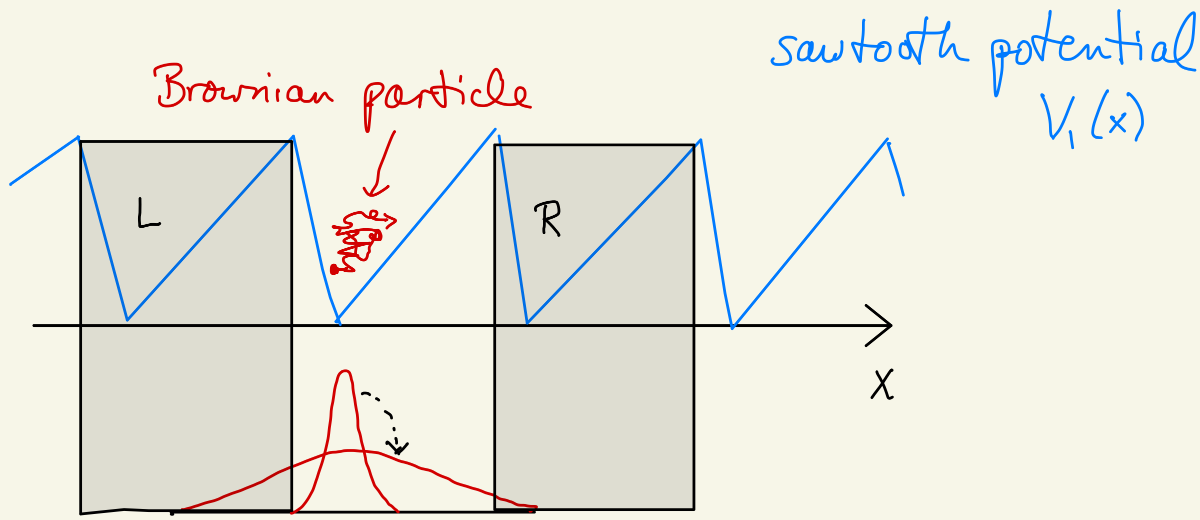
The ratchet & pawl (just like the paddle wheel) are subject to thermal noise, which will occasionally cause the pawl to move out of the way of the ratchet, allowing it to rotate CCW.

!! Fix the problem by suppressing the thermal fluctuations of the ratchet & pawl: $T_2 < T_1$.

Early to mid-1990's:

thermal / Brownian ratchet models

"rocking" ratchet, "flashing" ratchet, ...



Turn off potential $\longrightarrow$ particle diffuses.

Turn it back on $\longrightarrow$ particle more likely to get trapped in L than in R

"Toggle" back & forth between sawtooth potential $V_1(x)$ & flat potential $V_0(x) = 0$:

<u>net current to the left</u>.

“Feynman’s ratchet and pawl: an exactly solvable model”, *Phys Rev E* **59**, 6448-59 (1999). [1]

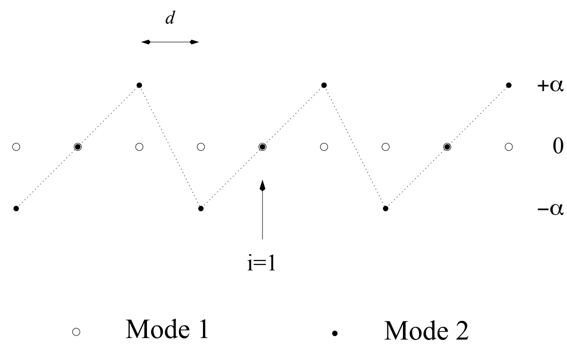


FIG. 1: Energy landscape for modes 1 and 2. Transitions between the two modes are driven by thermal fluctuations at temperature T_A, while leftward and rightward steps along the lattice are driven by fluctuations at temperature T_B.

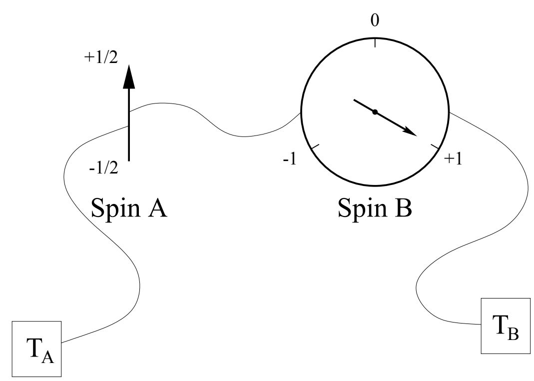


FIG. 2: Equivalent system: a spin-1/2 particle is coupled to a spin-1 particle, with the latter depicted by an arrow that can point in one of three directions on the face of a dial.

State energies:

State (n)	S_A	S_B	$E(S_A, S_B)$
1	-1/2	-1	0
2	-1/2	0	0
3	-1/2	+1	0
4	+1/2	-1	$-\alpha$
5	+1/2	0	0
6	+1/2	+1	$+\alpha$

[1] Figures used with explicit permission from the first author.

Transition rules: with probability rate Γ, the system attempts to change its mode (spin-1/2 flip); and with probability rate Γ, the system attempts to move $\pm d$ along the lattice (spin-1 rotation). In either case the move is accepted with probability

$$P_{\text{acc}} = \min\{1\,,\, \exp(-\Delta E/T_x)\} \qquad (x = A, B) \tag{1}$$

where ΔE is the change in the system's energy associated with the attempted move (*Metropolis rule*). The corresponding transition rate matrix is

$$\mathcal{R} \;=\; \Gamma \begin{pmatrix} -2 & \frac{1}{2} & \frac{1}{2} & \mu & 0 & 0 \\ \frac{1}{2} & -2 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1-\mu & 0 & 0 & 1 \\ 1 & 0 & 0 & -\mu - \frac{\nu+\nu^2}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{\nu}{2} & -\frac{3+\nu}{2} & \frac{1}{2} \\ 0 & 0 & \mu & \frac{\nu^2}{2} & \frac{\nu}{2} & -2 \end{pmatrix}, \tag{2}$$

with $\mu = e^{-\alpha/T_A}$ and $\nu = e^{-\alpha/T_B}$.

The system reaches a stationary state in which the average velocity of the particle along the lattice is:

$v > 0$
when $T_B > T_A$

$$v = -3d\frac{\Gamma}{N}(\mu-\nu)(1-\nu)(3\mu+4) \tag{3}$$

where $N(\mu,\nu) > 0$ is a fourth-order polynomial. In this stationary state the net flow of energy from reservoir A to reservoir B is:

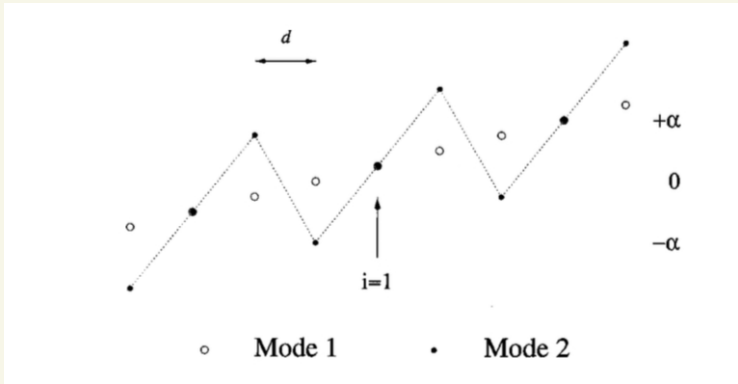
$$\dot{Q}_{A\to B} = 3\frac{\alpha\Gamma}{N}(\mu-\nu)P(\mu,\nu) \tag{4}$$

with $P(\mu,\nu) = 4 + 14\mu + 15\nu + 4\mu\nu + 5\nu^2 > 0$. The net rate of entropy production is

$$\dot{S} = \left(\frac{1}{T_B} - \frac{1}{T_A}\right)\dot{Q}_{A\to B} = 3\frac{\Gamma}{N}\left(\ln\frac{\mu}{\nu}\right)(\mu-\nu)P(\mu,\nu) \geq 0. \tag{5}$$

The model can be generalized to include an external load, and remains exactly solvable in this case. In the presence of this load, the model can act as a heat engine, or as a refrigerator, and its behavior in the linear response regime can be analyzed.

Now add an external load, f: (pulling leftward)



slope $= f$

For $T_B > T_A$ we have $v > 0$ when $f = 0$.

Thus for sufficently small $f > 0$ we expect to still have $v > 0$, which means the particle performs work against this load, i.e. it works as a <u>heat engine</u>:

$$\dot{Q}_{B\to A} > 0 \quad , \quad fv > 0 \;^{*} \quad (\sim \text{lifing a flea})$$

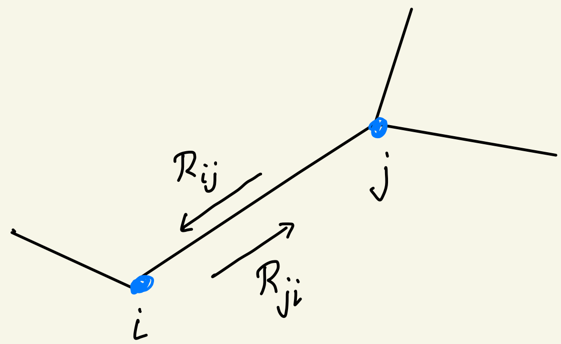
The model can also work as a <u>refrigerator</u>:

$$\dot{Q}_{B\to A} < 0 \quad , \quad fv < 0 \;^{*}$$

or as a <u>dud</u>: $\dot{Q}_{B\to A} > 0$, $fv < 0$

* these are examples of free energy transduction

Metropolis rule.



detailed balance: $\dfrac{R_{ij}}{R_{ji}} = e^{-\beta(E_i - E_j)}$

$$\begin{cases} R_{ij} = \Gamma \cdot P_{acc}(j \to i) \\ R_{ji} = \Gamma \cdot P_{acc}(i \to j) \end{cases}$$

When in state i:

Γ = rate of attempted transitions to j

$P_{acc}(i \to j)$ = probability that the attempt is accepted

Metropolis rule $\longrightarrow$ $= \min\{ \underset{E_j < E_i}{1}, \underset{E_j > E_i}{e^{-\beta \Delta E_{i\to j}}} \}$

$$\frac{R_{ij}}{R_{ji}} = \frac{P_{acc}(j \to i)}{P_{acc}(i \to j)} = e^{-\beta(E_i - E_j)}$$ ... confirm!

Random transitions among a network of states can also be used to model processes with variable #'s of particles.

Simplest example: <u>birth-death process</u>

P = a protein molecule that can exist in multiple copies, in a single cell

α = expression rate
= probability per unit time that a new copy of P is created in the cell

β = death rate
= prob. per unit time for a given copy of P to be destroyed

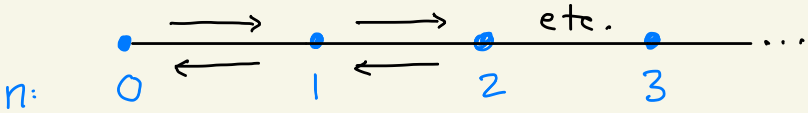
For 1 realization (i.e. one cell), let $n(t)$ denote the # of copies of P @ time t.

ensemble picture: $p_n(t)$ = prob. to find N copies @ time t

Birth-death processes, ctd.

α = expression rate, β = death rate

state space : $n \in \{0, 1, 2, \cdots\}$



$$\begin{cases} R_{nm} = 0 \quad \text{if} \quad |n-m| > 1 \\ R_{n+1,n} = R(n \to n+1) = \alpha \\ R_{n,n+1} = R(n+1 \to n) = (n+1)\beta \end{cases}$$

$$\frac{d\vec{p}}{dt} = R\vec{p} \quad , \quad R = \begin{pmatrix} - & \beta & 0 & 0 \\ \alpha & - & 2\beta & 0 \\ 0 & \alpha & - & 3\beta \\ 0 & 0 & \alpha & \ddots \end{pmatrix}$$

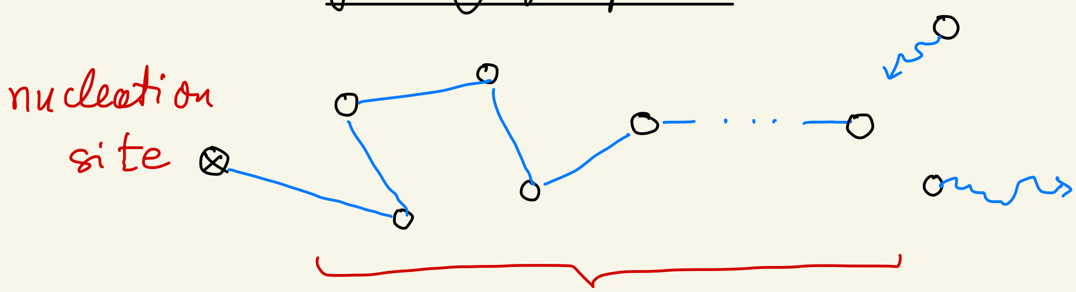
$$R\vec{\pi} = \vec{0} \longrightarrow \frac{\pi_{n+1}}{\pi_n} = \frac{1}{n+1}\frac{\alpha}{\beta} \equiv \frac{k}{n+1} \qquad \left(k \equiv \frac{\alpha}{\beta}\right)$$

$$\boxed{\pi_n = \frac{k^n}{n!}e^{-k}} \quad , \quad \langle n \rangle = k = \frac{\alpha}{\beta}$$

unique stationary state

Let's look @ a closely related example :

<u>growing polymer</u>



n = length (# of monomers)

α = single site attachment rate

β = single site detachment rate

$$R_{n+1,n} = \alpha \quad , \quad R_{n,n+1} = \beta$$

$$\mathcal{R} = \begin{pmatrix} - & \beta & 0 & 0 \\ \alpha & - & \beta & 0 \\ 0 & \alpha & - & \beta \\ 0 & 0 & \alpha & \ddots \end{pmatrix}$$

if $k < 1$

$$\mathcal{R}\vec{\pi} = \vec{0} \rightarrow \boxed{\pi_n = (1-k)\,k^n} \quad , \quad k = \frac{\alpha}{\beta}$$

If $k \geq 1$ $(\alpha \geq \beta)$, there is no stationary solution ... the polymer grows forever.